(7) $$\mathbf{v}=\nabla U\times\nabla V.$$

A necessary and sufficient condition that a vector function be solenoidal is that it admit of representation as the vector product of two potential vectors.

It should be noted that, when $\mathbf{v}$ is given, U and V are not uniquely determined, so that further conditions may perhaps conveniently be imposed upon them in special cases.

THE POSSIBLE ABSTRACT GROUPS OF THE TEN ORDERS 1909 — 1919.

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The real essence of fundamental theorems is frequently exhibited most forcibly by means of illustrative examples, especially when these examples have other elements of interest. The determination of all the possible abstract groups whose orders are equal to the numbers of the ten years 1909—1919 offers numerous instructive illustrations of important theorems, and exhibits some properties of these numbers which are at least of temporary interest.

Since 1909=23.83 is the product of two distinct primes such that the larger diminished by unity is not divisible by the smaller, it results that *the cyclic group of order* 1909 *is the only possible group of this order*. That is, there is only one group whose order is equal to the number of the present year. On the contrary, 1910=2.5.191 is the product of three distinct primes and hence every group of order 1910 contains an invariant subgroup of order 191 and also an invariant subgroup of order 955.* The latter may be either cyclic or non-cyclic, since 191—1 is divisible by 5.

As the group of isomorphisms of the cylic group of order 955 involves three operators of order 2 and the identity, there are four groups of order 1910 which involve a cyclic subgroup of order 955. The group of isomorphisms of the non-cyclic group of order 955 is the holomorph of the group of order 191 and hence it contains only one set of conjugate operators of order 2. When the invariant subgroup of order 955 is non-cyclic an operator of order 2 in the entire group must either transform each operator of this invari nt subgroup into itself or it must transform these operators according to an operator of order 2 in the group of isomorphisms. Hence there are two groups of order 1910 involving a non-cyclic subgroup of the order 955 and *there are exactly six distinct groups of order* 1910; *four of them involve a cyclic subgroup of order* 955 *while the remaining two do not have this property.*

*Cf. Burnside's *Theory of Groups of Finite Order*, 1897, p. 353.

The number $1911=3.7^2.13$ involves four prime factors but these factors are of such a nature that it is comparatively easy to determine all the possible groups of this order. Since 1 is the only divisor of 1911 which is of the form $13k+1$ it results from Sylow's theorem that a group of order 1911 can involve only one subgroup of order 13, and for similar reasons it can involve only one subgroup of order 49. Hence every group of order 1911 contains an invariant abelian subgroup of order 637. There are exactly five groups of order 1911 containing a cyclic subgroup of order 637 since the group of isomorphisms of this cyclic group involves four subgroups of order 3 and the identity. It is not difficult to see that there are ten groups of order 1911 containing the non-cyclic group of order 637, and hence *there are exactly fifteen distinct groups of order* 1911; *two of them are abelian, while the remaining thirteen are non-abelian.*

Every group of order $1912=3^3.239$ contains an invariant subgroup of order 239. If a subgroup of order 8 is also invariant the entire group is the direct product of the group of order 239 and one of the five groups of order 8. Hence there are exactly five groups of order 1912 such that each involves only one subgroup of order 8. If the subgroups of order 8 are not invariant there must be 239 such subgroups having four common operators, since the group of isomorphisms of the group of order 239 does not involve a subgroup of order 4. There are four possible groups in which these common operators form a cyclic group and three in which they form a non-cyclic group. Hence *the total number of abstract groups of order* 1912 *is* 12. Since 1913 is a prime *there is only one group of order* 1913.

The next number, $1914=2.3.11.29$, is the product of distinct primes and hence every group of this order involves an invariant subgroup of each of the orders 29, 29.11, 29.13.3. Each of these invariant subgroups is cyclic* and an operator of order 2 may transform the operators of orders 29, 11, and 3, either into themselves or into their inverses. Hence an operator of order 2 may transform the cyclic subgroup of order 957 in eight distinct ways and *there are exactly eight groups of order* 1914. Since $1915=5.383$ and $383-1$ is not divisible by 5 *there is only one group of order* 1915. *There are only four groups of order* $1916=2^2.479$ since $479-1$ is not divisible by 4.

A group of order $1917=3^3.71$ must be the direct product of the subgroups of order 71 and 27 respectively since $71\equiv 2 \bmod. 3$. Hence there are as many distinct groups of order 1917 as there are distinct groups of order 27; that is, *the number of abstract groups of order* 1917 *is* 5. A group of order $1918=2.7.137$ contains an invariant cyclic subgroup of order 7.137 and hence *there are four groups of order* 1918. It is clear that *there is only one group of order* $1919=19.101$. The next number, $1920=2^7.3.5$, lies outside the ten numbers under consideration but it may be remarked that it would be very much more difficult to determine the groups of this one order than it was to determine the groups of the ten orders under consideration. The

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year 1920 is so far ahead that it is to be hoped that before it arrives group theory may have made sufficient progress to determine all the groups of this order by means of general theorems. It need scarcely be added that most of the results given above are special cases of general theorems which were not mentioned in every case, since the direct proofs are so evident.

NOTE ON THE GENERAL QUARTIC.

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The general quartic equation

$$(1) \qquad \phi(x) \equiv a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

where the coefficients are all real, can be reduced to the form

$$(2) \qquad \phi(x) \equiv ax^4+bx^2+c=0,$$

by the following method.

Suppose the quartic resolved into the factors:

$$(a_1x^2+b_1x+c_1) \text{ and } (a_2x^2+b_2x+c_2).$$

Then effect the rational bilinear transformation $x=\frac{\alpha y+\beta}{y+1}$, obtaining

$$\phi(x)=\left[\frac{a_1(\alpha y+\beta)^2}{(y+1)^2}+\frac{b_1(\alpha y+\beta)}{y+1}+c_1\right]\left[\frac{a_2(\alpha y+\beta)^2}{(y+1)^2}+\frac{b_2(\alpha y+\beta)}{y+1}+c_2\right]=0.$$

In this expression α and β may be chosen so that the coefficients of the first powers of y shall be zero, after clearing of fractions.

The values of α and β fulfilling this condition are easily found to depend upon

$$(3) \qquad \alpha\beta=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \qquad (4) \qquad \alpha+\beta=2\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}.$$

That α and β are real, is determined from the equation